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Mathematics belongs to the humanities or

Preventing mathematics injury through dialogue

Translated from „Mathematik als Geisteswissenschaft. Der Mathematikschädigung dialogisch vorbeugen" (Gallin 2011) by Charles Gunn

Abstract

Mathematics is traditionally classified as one of the sciences. On the other hand, a consideration of its central role in the school curriculum provides compelling reasons to count it among the humanities, for here it is concerned essentially with the individual human development of the students. Just as in bodily development, also in such inner development it should be kept in mind that just because a teaching method is available at a given stage of development doesn't mean it should be allowed. The goal is to support the inner ripening while avoiding collateral damage. A proven strategy here is for teachers to place more confidence in their students. Lack of confidence appears in traditional teaching methods in four typical forms: in the transfer of knowledge, in the application of algorithms, in the production of exercises, and in the use of modern visualization tools like 3D models, computer-aided animations, interactive whiteboards, etc. Dialogic learning, in which understanding stands at the centre of mathematics instruction, provides a framework in which increased confidence in the students can be put into practice, allowing the potential for inner development to unfold.

Originally I wanted to be an architect, following the examples of my father and my grandfather. That's why, as a nine-year-old child, I drew plans of houses as I often saw them. A worn-out wooden triangle with one right angle and two half right angles was the only design tool I possessed at the time; with it I set to work. First, of course, there was a rectangle to draw as the floor plan of the house: One side, a right angle, the second side, again a right angle, the third side in the same length as the first and then the third right angle, whose second leg gave the fourth side of the rectangle. But alas: the fourth angle was not right and the lengths of the second and fourth side were also not equal. Already, in drawing the rectangle, I had failed, and immediately christened my "crooked house" with the fantasy name "Chesa Krümlas" (Fig. 1). My expectation that a quadrilateral with three right angles would necessarily have a fourth right angle was severely disappointed by the reality of my awkward constructions. And in my impatience I did not want to hear the advice of my older sisters regarding the use of a so-called T-square and its reliable parallel shifting. My little table was in any case too small for the big instrument, which would only have been available in my father's office.

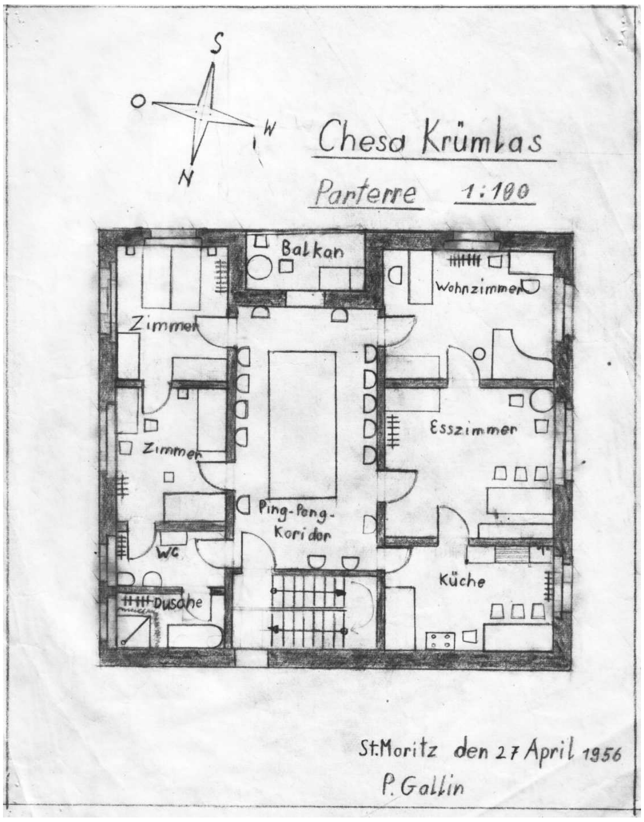

Fig. 1: First attempt to draw a floor plan.

Only in high school geometry class did I begin to realize that there is always a gap between reality and mathematical idealization, no matter how precise one tries to be. Euclid's ideas about dimensionless points and infinitely thin straight lines stretched out in this or that direction along with the well-behaved families of parallels led me to change my career aspirations from architect, who has to struggle with reality, to mathematician, who can busy himself with perfect, self-consistent ideas. Thus, I believed that mathematics finds its sole purpose in describing reality with idealizations, and that, consequently, investigations in idealizations ought to be a convenient way of studying nature. With the beginning of physics classes, this view was confirmed, so that in short order I equated mathematics and science. How shocked I was, however, when my math teacher said, wagging his finger at me, "No, Peter, mathematics belongs much more to the humanities!" Perhaps it was this shock that led me to study physics – of course, naturally, theoretical physics.

In the course of my studies and my teaching internship afterwards, it became more and more apparent that the human mind and spirit interested me as much as the functioning of nature. I was particularly fascinated by how people responded emotionally and intellectually to unexpected mathematical-scientific phenomena. This interest in the interaction between the inner human being and the outer expression of a problem clearly pointed me towards the teaching profession. And because for me the mental mastery of a problem carried more weight than skilful experimental procedure, I returned to mathematics and began my teaching in high school in this subject. Certainly at the beginning there were still clear traces of a physics-oriented past, so that – like many young graduates – I assigned almost too much importance to the so-called "everyday relevance" of mathematics. Only contact with real people and their needs in learning and comprehending mathematics in the high school classroom changed my picture of mathematics from that of an auxiliary science for the mastery of nature and technology via a self-consistent, abstract science of relationships and patterns independent of all external supports into a field of activity in which people push themselves to the limit of their capacities and thereby experience a validation of themselves. Descartes' famous quote "cogito, ergo sum" became for me the pedagogical guideline for mathematics lessons, by reformulating it as "I engage in mathematics, therefore I sense myself"[1]. It was only with this image of mathematics as a challenge to the human spirit that I was able to recognize and explain why the subject of mathematics has gained such an important position in the canon of school subjects: at its centre stands an education and training mission only implicitly present in the prefaces of the syllabi.

In more sober terms, we could describe this goal as a comprehensive competence to act, as defined by Franz E. Weinert (Weinert 2001, p. 51): The competence to act consists of the necessary conditions, capable of development, that enable a person in one specific field of action (as specialist or professional) to operate successfully (Ruf 2008, p. 246). The crucial factor in this concept is that the competence to act exhibits three necessary aspects. In the centre is the technical aspect with the corresponding knowledge and skills. It is this aspect that can be studied in the tests in use today (such as PISA). However, the technical aspect is embedded in the social aspect of action competence, where abilities for changing perspectives, for cooperation, for mutual appreciation, and for dialogue are called for. Finally, the two aspects are embedded in the third, the personal aspect of action competence, whose ingredients include the capacity to reflect, motivation, will, self-concept, values, and the question of meaning. It is only in the successful interaction of all three aspects that one can speak of achieved competence. It is generally accepted that the second and third aspects can not be tested. They can in fact only be recognized and cultivated in school through an accompanying teacher.

[1] „Vom Sinn des Mathematikunterrichts" is the name of an article that I wrote earlier on this theme (Gallin 2002).

Against the background of my personal teaching experience and the educational-scientific concept of "competence to act", mathematics has become for me a discipline that revolves around the development of the human individuality, hence ought to be counted among the humanities. Since I am interested in mathematics as a school subject and its methodological aspects, I will in no way take sides in the theoretical dispute about the position of mathematics as a university department.[2] In fact, in this context there is considerable controversy whether to consider mathematics as one of the humanities, but there has been virtually no discussion of its position in the school. My experience as a mathematics teacher over the years has shown me that mathematics education benefits from considering it pragmatically as almost purely "humanistic" rather than "scientific". Instruction becomes only purged and simplified, but also comes closer to the students. Hans Freudenthal wrote in his article "Mathematics – an inner attitude" (Freudenthal 1982): "What's always true is: The student acquires mathematics as an inner state of mind only by relying on his own experience and his own understanding."

There is a danger that deep-seated preconceptions regarding mathematics derived from its "scientific" aspect will exert unhealthy influence in this "humanistic" setting. This appears for example in the belief that the student of mathematics should be rigorously trained so that he/she is able to function quickly and without errors and hence to solve fixed abstract tasks with a handy set of recipes. Rather, the actual goal of the instruction is learning to recognize and understand connections; as a result, efficiently solving tasks is reduced, as it were, to a welcome side-effect. Thus, the central meaning of the active person as a totality becomes clear: as soon as insight and understanding come into play, it is no longer sufficient to only promote the subject-specific competences of the learner, but one also has – as pointed out by Weinert – to pay attention to their motivational and emotional foundations.

What are the concrete consequences for the design of math instruction? The philosopher Hans-Georg Gadamer, who thought a lot about the central question of understanding, provides a first indication (Gadamer 1959): "The starting point for any understanding is that something appeals to us: that is the paramount hermeneutic condition." This is an expression of the fact that the motivational aspect of every specialized knowledge is present from the start. The physicist Martin Wagenschein takes this advice one step further and emphasizes not only the motivational but also the social aspect (Wagenschein 1986): "Conversation brings us real understanding. Starting and stimulated by something mysterious, in search of what is fundamental." As with Gadamer, the starting point is a single person touched by a "riddle", but here the single perspective is supplemented via exchanges with other persons who have thought about the same riddle.

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[2] In his lecture „Mathematik als Kulturleistung [Mathematics as cultural achievement]" of November 28, 2008, Christoph Schweigert shows in detail why mathematics belongs to the humanities. Download-Links (March 27, 2010):
www.math.uni-hamburg.de/home/schweigert/2008.geist.pdf
www.math.uni-hamburg.de/home/schweigert/transp.html

As plausible and simple as these insights and recommendations are, they are regularly and consistently disregarded in traditional teaching methods. This is not due to carelessness in daily teaching practice, but first and foremost carefully thought-out textbooks and teaching concepts. So it’s not merely a question of small adjustments, for example, in the speed of learning. The intellectual development of our students is at stake, depending on whether the given pedagogical foundations provide them with the possibility of understanding – or not. Of course, gifted students have always taken the liberty to pose problems, to think further, to discuss and thus to advance to a real understanding. Weaker students, however, are misled by traditional teaching methods to content themselves with a superficial imitation of authentic learning. And because the teacher also has to focus on the weaker students, the fateful paradigm has emerged that exactly these students need easy-to-use recipes to achieve at least minimal (technical) skills. A vicious cycle arises: teachers, supported by textbooks and teaching aids, respond to the effort learners make by subjecting the mathematical procedures to an increasingly sophisticated and clever subdivision in the belief that very simple and small steps ought to be understandable by all. Exactly this is a deception, because through the many small steps, through the so-called segmentation (Gallin & Ruf 1990, p 36 et seq.), learners are removed even further from the actual theme, they lose both the insight and the overview, and feel even less “spoken to” personally. The necessary condition for understanding is missing. The learners become completely dependent on given explanatory schemes.

A vicious circle can only be broken with one fell swoop. This means that many “self-evident” aspects of traditional teaching methods must be questioned and perhaps even jettisoned. Could it be that our teaching efforts have a detrimental effect on the mental development of our children, that they cause the so-called “mathematics injury”? Freudenthal says in the above-mentioned article (Freudenthal 1982): “Yes, maybe useless mathematics learning in fact has caused damage.” So where should we focus our attention? To answer such questions I would like to pick out the relevant aspects from the concept of “dialogic learning”, which offers a comprehensive educational solution (Ruf & Gallin 2005), and then discuss them specifically in the context of mathematics instruction.

Confidence

The main problem of traditional mathematics teaching methods is that teachers usually do not show enough confidence in the learners. This doesn’t happen maliciously, but out of a striving to give the learners only the best. This is particularly evident in four areas, where the prevailing opinion is:

- that you should offer the collected technical knowledge, already neatly segmented, via a meticulously-prepared transfer process,
- that you should teach faultlessly functioning algorithms,
- that you should generate perfect exercises for learners, and
- that you should provide only the best possible illustrations and figures.

The lack of confidence in these four areas finds expression as follows:

1. The separation between imparting knowledge and practicing it usually takes this form: the theory is presented in classroom instruction itself and the exercises are given as homework. However, giving out suitable assignments in the form of simple but real problems of the discipline and research questions as homework rather than ordinary exercises may well break down this traditional distinction. What is crucial is that the learners can – indeed, must – struggle with the central issues for a sufficiently long time. Whether this happens at home or at school is not so important. As an example consider the calculation of powers with general exponent. Normally, the students already know calculation with natural exponents from the 7th or 8th school year. In later grades they are supposed to get to know negative and fractional exponents, even irrational exponents. Looking at the textbooks one might think that the teacher should discuss the definitions for these generalizations in the classroom and then consolidate the imparted knowledge with some homework exercises. The textbooks usually offer a lot of material, first for arithmetic with integer exponents, then for arithmetic with fractional exponents and also with irrational exponents. Finally, mixed examples and exercises are introduced. Experience shows, however, that even after several weeks of preparation, in the transition to the mixed tasks considerable difficulties emerge – it can really seem as if the individual sub-topics had not been treated at all. This experience has led me to introduce the subject very differently and to present learners the central assignment right at the beginning via the following question: How can we define the as-yet unknown numbers 2^{-3} on the one hand and $2^{1/3}$ on the other if we make the assumption that the familiar laws of exponents, in particular $2^a \cdot 2^b = 2^{a+b}$ and $(2^a)^b = 2^{ab}$, remain valid? It turns out that learners are quite able to deal with such "research questions" and come up with answers. That is to say, they can be trusted with much more than they usually are. This assignment also reveals that segmentation into first negative and then fractional exponents is unnecessary. In short: you start with "mixed tasks" instead of spending a lot of time on the pieces.

2. Another form of lack of confidence, more subtle because it's hidden and unconscious, can be traced back to the usual mathematical way of presenting results. Definitions and theorems dominate the mathematical literature. This encourages teachers to provide students with puncture- and rip-proof definitions and algorithms for specific problems, such as the fractional division rule "to divide by a fraction, you ..." or the equivalence "$\log_a(b) = c \Leftrightarrow a^c = b$", to define the logarithm, or also the Gaussian elimination algorithm to solve a system of linear equations. For the first rule all you need is the core idea "dividing by half gives more" along with an illustrative story (Ruf & Gallin 2005, Volume 2, p. 25). The core idea for the logarithm, so little loved by students, is equally succinct: "The logarithm in base *a* of *b* is the exponent you have to set atop *a* in order to obtain *b*." With this admittedly very sloppily formulated theorem the learners are empowered to imagine a concrete meaning for the logarithm and in fact to discover on their own the laws of logarithms (including the little-known "eating-up" laws $\log_a(a^b) = b$ and $a^{\log_a(b)} = b$) and other

formulations. For systems of linear equations I want to go into more detail. Normally one introduces linear equations with one variable (unknown). The corresponding word problems are also supposed to be mastered using one variable. However, experience shows that it would often be much more natural to start with two or more variables. The following exercise is taken from "Algebra 1" (Deller et. al. 2000, Exercise 131, p. 67) and is intended to be solved using one variable: "Peter brings red wine, white wine, and mineral water out of the cellar. He brings the same number of wine bottles as water bottles and twice as many white wine as red wine bottles, 30 bottles in all. How many bottles of each sort does he bring up?" Here it's natural to declare the variables r, w and m as the numbers of the corresponding bottles, and then write down three equations for the three unknowns. The resulting elimination using the substitution rule doesn't offer the learners any difficulty, in fact, they discover it themselves. How much more difficult, on the other hand, is it to translate this exercise using only one variable (for example, x = number of red wine bottles) directly in an equation! The equation $x + 2x + (x + 2x) = 30$ can only be easily written down when you have the inspiration to use x to represent the number of red wine bottles and to use it to express the number of the other bottles. In short: you have to mentally penetrate the exercise before you write down anything, to obtain the same result that you do with three variables only after writing down and calculating with the three equations. You can see in this example first of all that it doesn't make sense to spend a lot of time with one equation and one unknown, and secondly that it's possible to introduce very early on the process of elimination involving several variables. For that to work, however, the learners don't have to memorize an algorithm with strict procedures leading to the so-called triangular form of the system, but rather they should follow their own path that can be described by a single core idea more or less as follows: "If you have three equations with three unknowns, then just make sure you arrive at two equations with two unknowns by applying the addition procedure in two different ways." More doesn't need to be said. You can trust the learners to discover the process on the basis of the concrete example. And later, in a higher grade, you can even assign the task to develop a schema that doesn't depend on the particular fortunate constellation of the given coefficients in the system of equations. Even the Gaussian elimination algorithm can be found in this way by the learners.

3. Math teachers spend a lot of time to produce exercises, both for practice and for tests, that are well-adapted to the theory, are nicely graduated from simple to difficult, and furthermore yield "nice numbers" in interim calculations and the final result. (Malicious tongues even claim that math teachers are so good at math only because they are constantly producing these exercises.) But this overlooks the fact that exactly in this domain you can expect very much from the learners, they can discover astounding and surprising things. The production of exercises for their fellow students carries an enormous potential of motivational energy. For one thing, the initial phase of actual invention and production is highly motivating, when the goal consists of generating on the one hand a solvable and on the other hand a genuinely difficult exercise. Furthermore, the solution of such an exercise is also particularly

attractive and exciting, since the author is a personal acquaintance and can, if need be, be consulted in the case of unsolvable or overly difficult exercises. The social aspect of the competence to act comes into play in a very natural way. You can read more about this in the second volume of „Dialogisches Lernen in Sprache und Mathematik" (Ruf & Gallin 2005, p. 167 ff.).

4. Probably the most subtle form of lack of confidence occurs with the best of intentions when we use visual aids, whether it be models, such as the plexiglass ones that can be found in the collections of math departments, or whether it is modern, computer-aided animations on monitors, projection panels, and interactive whiteboards. Caution is particularly called-for with these seductively beautiful and efficient achievements since here the inner development of the learners is especially at risk. The "trust" constellation here verges on the paradoxical: on the one hand one expects the learners to receptively digest the smooth and perfect visualizations, while on the other hand one does not trust them to productively develop for themselves the corresponding mental pictures, drawings, even models. The inner development in this case is not tied to a particular age, but rather to the problem itself. In relation to a technical question or mathematical problem, everyone starts out as a child, so to speak. Through activity focused on the problem setting, the person ripens gradually, in that they ask new questions, possible approaches come to mind, and ideas appear. They work on the problem with more or less success, generally until the point arrives at which they stand up and want to enter into an exchange with others. This is a ripening process that cannot be shortened and that is irreplaceable for an understanding of the mathematical situation, as we know from Gadamer, Wagenschein, and Weinert. Whether or not the desired goal is attained, doesn't play an important role. Someone who, for example, talks about a regular dodecahedron and believes they have to describe its net to the children, acts reprehensibly, first in that they don't trust the children to produce it themselves and secondly, they injure the children by trying to shortcut their inner development by jumping ahead[3]. With external visualizations we need – polemically expressed – a didactic rating system to protect young people, whereby the criterion is less their actual age than their relation to a particular math problem. With inner ripening processes it's just like with bodily ones, what you present to young people has to match their actual state of development. If you share a visualization too soon to an unprepared, unripe person, then it injures them. If, however, the ripening is completed to a certain degree, then these visualizations can represent a genuine pleasure.[4] Finding the right measure and the right moment here is a real art. Wrapping up: models, visualizations, and

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[3] See the example at the end of this article.

[4] It would be an over-reaction, as in the iconoclasm of the Reformation in the 16th century, to completely banish pictorial aids and visualizations from the math classroom. (Citations from Wikipedia March 27, 2010: Following the orders of reformation theologians and newly-converted rulers, paintings, sculptures, stained-glass windows and other works with visual representations of Christ and the saints as well as additional church decorations, occasionally also church organs, were removed from the churches. https://en.wikipedia.org/wiki/Iconoclasm)

animations are well and good, but they can still be X-rated for children. Bringing these reflections in connection with the previous paragraph, a rating system should also be applied to theorems and algorithms, with which the children should only be confronted after they have attained a certain inner independence (Gallin 2002).

Let’s illustrate through an example how close to each other visualizations can be that are, on the one hand, damaging and, on the other hand, supportive for inner development. Imagine that you cut a thick, oblique slice off a cylindrical sausage and then peel the skin off this slice. How does the resulting flattened-out skin look like? This gives rise to a whole series of mathematical questions including how to prove the answer found to the first question. (We remark in passing that this line of inquiry has a very practical domain of application: when a metalworker wants to correctly cut tin to construct two cylindrical pipes that intersect at right angles, he has to solve exactly this problem.) That’s all that needs to be said about the supportive framing. We turn now to an older book (Steinhaus 1950, S. 198, 199) containing a picture sequence presenting the same mathematical content: a cylindrical candle is wrapped around several turns with a wide strip of paper. Then it’s cut in two in the middle of the paper wrapping at an oblique angle. If you now unroll one of the two halves of the cut-through paper, an interesting plane curve produced by the knife-cut shows itself. Why is this framing injurious in comparison to the first? Ignoring the fact that normally, in contrast to sausages, you don’t cut candles, the periodicity of this interesting curve is revealed through the repeated wrapping with the paper. This leaves no room for inner construction of the periodicity – and differentiability – of the curve. The magic spell of the riddle is thereby broken; which can furthermore lead to mathematical injury in that it holds the learner back from their own inner activity and brings instead an external didactic arrangement into the foreground.

Recently a new approach to regular mathematics instruction has appeared that explicitly trains the individual power of “imagining” of the student. This research work of Christof Weber is concerned with what he calls “exercises in mathematical imagining”. In his books and articles (Weber 2007, 2008, 2009 and 2010) there are many examples of how, just by inner mental picturing, learners can penetrate a particular mathematical problem-setting in astounding depth. With his unpretentious starting phrase, “imagine to yourself …” the door in the math lesson is opened in the simplest way to the inner development of the learners, one that works on every instruction level in the school. That’s even valid for university students of mathematics, who by means of an “imagining exercise” devoted to the spheres of Dandelin are enabled subsequently to sketch a spatially convincing picture of the configuration, without ever having seen a corresponding real model, which of course do exist. As soon however as they have constructed the inner mental pictures, they experience the real model as a wonderful confirmation of their accomplishment and their maturity. And they remember the phenomenon of the spheres of Dandelin even years after the imagining exercise. Plato had already in his work repeatedly referred to the “inner eye” (often translated as “eye of the soul” or “understanding”), that is more valuable than a thousand bodily eyes and with which alone the truth can be seen (Schleiermacher 1963). For him, occupying yourself with mathematics was nothing else than training this kind of seeing.[5]

Listening and caring for

Now that we have thoroughly discussed how the lack of confidence appears in traditional teaching methods, two subsequent teaching gestures naturally enter our field of view. In order for the teacher to know at all times the level of ripeness of their students, they must continually inspect what the students produce. The litmus test of confidence is that learners are encouraged to produce work that would be considered impossible in traditional lessons. It is therefore all the more important that the teacher appreciates these often unpredictable achievements, evaluates them and lets them flow back into the lessons. This process can be described as "listening", which expresses that the primary task of the teacher is shifting from production to reception, while conversely learners move from reception to production (Ruf & Gallin 2005, vol. 2, p. 10 ff.). Actually, this reversal has long been recognized: Producing is easier than receiving, because reception always involves two points of view, while in production only one's own point of view counts, at least at first. Hence, learners should be allowed to begin by being productive.[6] However, progress can only be made if they are aware that their productions are being taken seriously and that they have a direct influence on what is happening in the classroom, so it is imperative that the teacher collects and reviews all the students' work. More can be found on this practice of dialogic learning in several publications (Gallin 2008, 2009).

A simple calculation confirms that looking through the student productions involves a significant amount of work for the teacher. All the more reason to keep an eye on areas where work can be reduced or avoided. Here confidence plays a central role: if more is expected of the students, many tasks traditionally performed by the teacher disappear, especially in the area of preparing and distributing material. In addition to this compensation, which takes some getting used to, another benefit should not be underestimated: the teacher no longer needs to work artificially and with intense effort to cultivate a personal, empathetic and friendly relationship to the learners; out of trust and listening there develops for the learners a tangible sense of being "cared for", which comes through the subject itself and not through extracurricular activities. In this way they get to know the subject as something very personal in two ways: On the one hand, the teacher personifies the subject and demonstrates how the subject enables inner development even in this elementary, school-based version; on the

[5] While Plato's thoughts regarding the inner eye and mathematics are practical and relevant for school teaching, we can no longer follow his recommendations for the school itself. He never concerned himself with the education of the general population, but rather only with the elite (warrior class). His strictly hierarchical thinking is diametrically opposed to a public school.

[6] This principle is often applied in German lessons when children are introduced to reading and writing in first grade: „Lesen durch Schreiben" ("Reading by Writing"), as in Jürgen Reichen (Reichen 1988).

other hand, the person of the learner feels that with their own ideas, approaches and contributions they are not hopelessly mismatched when confronting the venerable subject of mathematics.

Finally, a concrete example from the fifth grade[7] shows how effortlessly and directly one can move in math instruction from trusting to listening to caring for: it concerns the topic "nets of solid bodies." A first assignment to the pupils might be put as follows: "In mathematics one calls a flat cut-out pattern for a solid body a 'net'. Draw some nets of bodies, which you already know." And the classroom quickly begins to percolate and the children bring forward their treasures, which are of course taken up and discussed. A second directive could be: "Here you see a body. Draw a net for it!" When children are used to retaining the traces of their mental steps pearls like Leon's are always to be found. He sketches the accompanying figure (Fig. 2) into his journal as a net for a right circular cone, but quickly notes his mistake and draws a correct figure. Luckily, the teacher has saved his first attempt from the eraser.

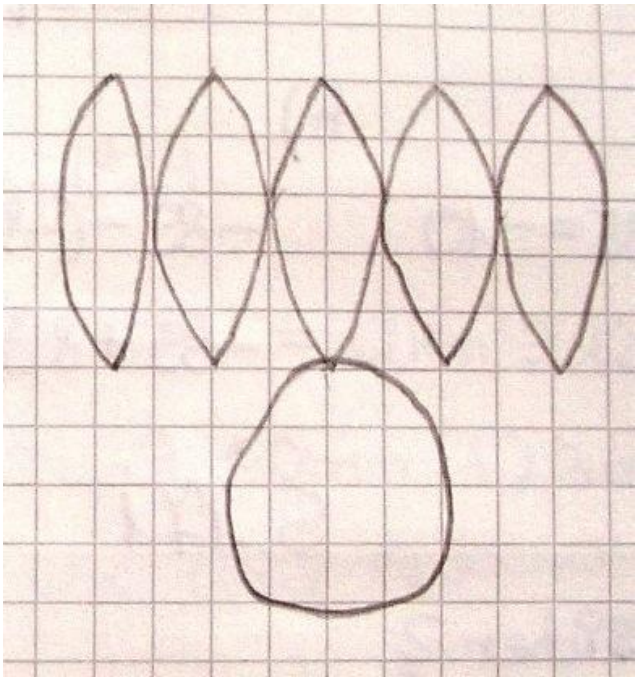

Fig. 2: Leon's first attempt at the net of a circular cone.

Which teacher would have had the fantasy and the courage to invent Leon's first net and present it as a riddle for the learners? Now however a lively discussion ensues regarding Leon's remarkable solid. The question arises at the end, what the net of a sphere or an oblique circular cone looks like. Questions that can also fluster teachers, since they bring the rectification of an ellipse into play. (It's astounding that in the

[7] The example from 2009 comes from a class taught by Maren Distel, Hegau-Gymnasium in Singen (Hohentwiel).

attempt to construct a net for a cylindrically segmented "sphere" one encounters the same curve already encountered in the experiment with the sliced sausage.) This example from the classroom shows what possibilities for inner development open up, as soon as you display confidence in the students rather than immediately presenting a methodological solution worked out to the last detail.

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NOTE: A collection of articles by Peter Gallin on dialogic learning, translated into English, is available for download at www.gallin.ch/_AuswahlEnglArtikelGallin.zip.

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